Discover Calculus

Single-Variable Calculus Topics with Motivating Activities

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Chapter 1

Limits

1.1 Introduction to Limits

Almost 2,500 years ago, the Greek philosopher Zeno of Elea gifted the world with a set of philosophical paradoxes that provide the foundation for how we will begin our study of calculus. Perhaps the most famous of Zeno's paradoxes is the paradox of Achilles and the Tortoise.

1.1.1 Achilles and the Tortoise

In the paradox of Achilles and the Tortoise, the Greek hero Achilles is in a race with a tortoise. Obviously the tortoise is much slower than Achilles, and so the tortoise gets a 100m head start. The race begins, and while the tortoise moves as quickly as it can, Achilles has the obvious advantage. They both are running at a constant speed, with Achilles running faster. Achilles runs 100m, catching up to the tortoise's starting point.

In the meantime, the tortoise has moved 2 meters. Achilles has almost caught up and passed the tortoise at this point! In a *very* short time, Achilles is able to run the 2 meters to catch up to where the tortoise was. Unfortunately, in that short amount of time, the tortoise has kept on moving, and is farther along by now.

Every time Achilles catches up to where the tortoise was, the tortoise has moved farther along, and Achilles has to keep catching up.

Can Achilles, the paragon of athleticism, ever catch the tortoise?

1.1.2 A Modern Retelling

A college student is excited about having enrolled in their first calculus class. On the first day of class, they head to class. When they enter the hallway with their classroom at the end, they take a breath and excitedly head to class.

In order to get to class, though, they have to travel halfway down the hallway. Almost there.

Now, to go the rest of the way, the student will half to get to the point that is halfway between them and the door. Getting closer.

They're getting excited. Finally, their first calculus class! But to get to the class, they have to reach the point halfway between them and the door.

Their smile starts fading. They repeat the process, and go halfway from their position to the door. They're closer, but not there yet.

If they keep having to reach the new halfway point, and that halfway point is never actually *at* the door, then will they ever get there?

Halfway to the door, then halfway again, closer and closer without ever getting there.

Will the student ever get there, or are they doomed to keep getting closer and closer without ever reaching the door?

1.2 The Definition of the Limit

1.2.1 Defining a Limit

Activity 1.2.1 Close or Not? We're going to try to think how we might define "close"-ness as a property, but, more importantly, we're going to try to realize the struggle of creating definitions in a mathematical context. We want our definition to be meaningful, precise, and useful, and those are hard goals to reach! Coming to some agreement on this is a particularly tricky task.

(a) For each of the following pairs of things, decide on which pairs you would classify as "close" to each other.

- You, right now, and the nearest city with a population of 1 million or higher
- Your two nostrils
- You and the door of the room you are in
- You and the person nearest you
- The floor of the room you are in and the ceiling of the room you are in

(b) For your classification of "close," what does "close" mean? Finish the sentence: A pair of objects are *close* to each other if...

(c) Let's think about how close two things would have to be in order to satisfy everyone's definition of "close." Pick two objects that you think everyone would agree are "close," if by "everyone" we meant:

- All of the people in the building you are in right now.
- All of the people in the city that you are in right now.
- All of the people in the country that you are in right now.
- Everyone, everywhere, all at once.

(d) Let's put ourselves into the context of functions and numbers. Consider the linear function $y = 4x - 1$. Our goal is to find some x-values that, when we put them into our function, give us y-value outputs that are "close" to the number 2. You get to define what close means.

First, evaluate $f(0)$ and $f(1)$. Are these y-values "close" to 2, in your definition of "close?"

(e) Pick five more, different, numbers that are "close" to 2 in your definition of "close." For each one, find the x-values that give you those y-values.

(f) How far away from $x = \frac{3}{4}$ can you go and still have y-value outputs that are "close" to 2?

To wrap this up, think about your points that you have: you have a list of x-coordinates that are clustered around $x = \frac{3}{4}$ where, when you evaluate $y = 4x - 1$ at those x-values, you get y-values that are "close" to 2. Great!

Do you think others will agree? Or do you think that other people might look at your list of y-values and decide that some of them *aren't* close to 2?

Do you think you would agree with other peoples' lists? Or you do think that you might look at other peoples' lists of y-values and decide that some of them *aren't* close to 2?

Definition 1.2.1 Limit of a Function. For the function $f(x)$ defined at all x-xalues around a (except maybe at $x = a$ itself), we say that the **limit of** $f(x)$ as x approaches a is L if $f(x)$ is arbitrarily close to the single, real number L whenever x is sufficiently close to, but not equal to, a. We write this as:

$$\lim_{x \to a} f(x) = L$$

or sometimes we write $f(x) \to L$ when $x \to a$. ◊

When we say "around $x = a$", we really just mean on either side of it. We can clarify if we want.

Definition 1.2.2 Left-Sided Limit. For the function $f(x)$ defined at all x-xalues around and less than a, we say that the **left-sided limit of** $f(x)$ as x approaches a is L if $f(x)$ is arbitrarily close to the single, real number L whenever x is sufficiently close to, but less than, a. We write this as:

$$\lim_{x \to a^-} f(x) = L$$

or sometimes we write $f(x) \to L$ when $x \to a^-$. ◊

Definition 1.2.3 Right-Sided Limit. For the function $f(x)$ defined at all x-xalues around and greater than a, we say that the **right-sided limit of** $f(x)$ as x approaches a is L if $f(x)$ is arbitrarily close to the single, real number L whenever x is sufficiently close to, but greater than, a. We write this as:

$$\lim_{x \to a^+} f(x) = L$$

or sometimes we write $f(x) \to L$ when $x \to a^+$. ◊

Theorem 1.2.4 Mismatched Limits. *For a function $f(x)$, if both $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then we say that $\lim_{x \to a} f(x)$ does not exist. That is, there is no single real number L that $f(x)$ is arbitrarily close to for x-values that are sufficiently close to, but not equal to, $x = a$.*

1.2.2 Approximating Limits Using Our New Definition

Activity 1.2.2 Approximating Limits. For each of the following graphs of functions, approximate the limit in question. When you do so, approximate the values of the relevant one-sided limits as well.

(a) Approximate $\lim_{x \to 1} f(x)$ using the graph of the function $f(x)$ below.

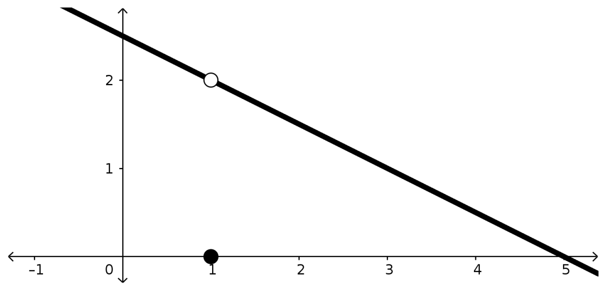


Figure 1.2.5

(b) Approximate $\lim_{x \to 2} g(x)$ using the graph of the function $g(x)$ below.

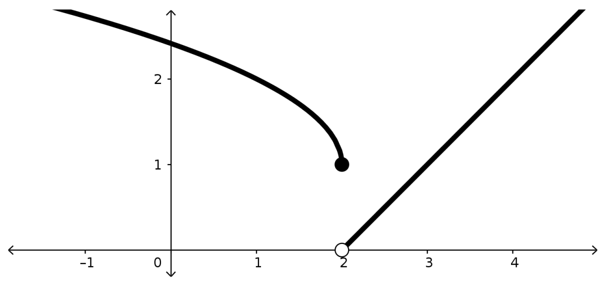


Figure 1.2.6

(c) Approximate the following three limits using the graph of the function $h(x)$ below.

- $\lim_{x \to -1} h(x)$
- $\lim_{x \to 0} h(x)$
- $\lim_{x \to 2} h(x)$

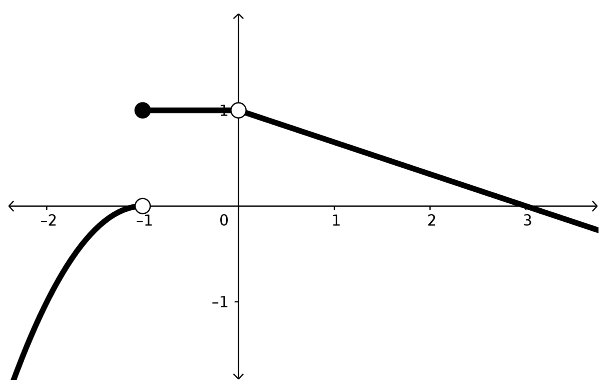


Figure 1.2.7

(d) Why do we say these are "approximations" or "estimations" of the limits we're interested in?

(e) Are there any limit statements that you made that you are 100% confident in? Which ones?

(f) Which limit statements are you least confident in? What about them makes them ones you aren't confident in?

(g) What extra details would you like to see to increase the confidence in your estimations? Are there changes we could make to the way these functions are represented that would make these approximations better or easier to make?

Activity 1.2.3 Approximating Limits Numerically. For each of the following tables of function values, approximate the limit in question. When you do so, approximate the values of the relevant one-sided limits as well.

(a) Approximate $\lim_{x \to 1} f(x)$ using the table of values of $f(x)$ below.

Table 1.2.8

x	0.5	0.9	0.99	1	1.01	1.1	1.5
$f(x)$	8.672	9.2	9.0001	-7	8.9998	9.5	7.59

(b) Approximate $\lim_{x \to -3} g(x)$ using the table of values of $g(x)$ below.

Table 1.2.9

x	-3.5	-3.1	-3.01	-3	-2.99	-2.9	-2.5
$g(x)$	-4.41	-3.89	-4.003	-4	7.035	2.06	-4.65

(c) Approximate $\lim_{x \to \pi} h(x)$ using the table of values of $h(x)$ below.

Table 1.2.10

x	3.1	3.14	3.141	π	3.142	3.15	3.2
$h(x)$	6	6	6	undefined	5.915	6.75	8.12

(d) Are you 100% confident about the existence (or lack of existence) of any of these limits?

(e) What extra details would you like to see to increase the confidence in your estimations? Are there changes we could make to the way these functions are represented that would make these approximations better or easier to make?

1.3 Evaluating Limits

1.3.1 Adding Precision to Our Estimations

Activity 1.3.1 From Estimating to Evaluating Limits (Part 1). Let's consider the following graphs of functions $f(x)$ and $g(x)$.

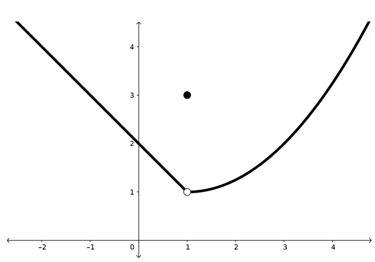


Figure 1.3.1 Graph of the function $f(x)$.

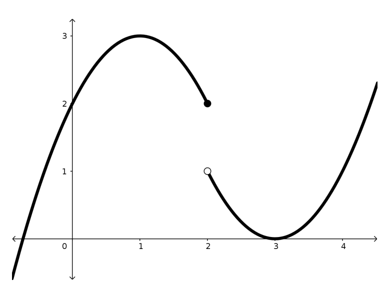


Figure 1.3.2 Graph of the function $g(x)$.

(a) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim_{x \to 1^-} f(x)$

- $\lim_{x\to 1^+} f(x)$
- $\lim_{x\to 1} f(x)$

(b) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim_{x\to 2^-} g(x)$
- $\lim_{x\to 2^+} g(x)$
- $\lim_{x\to 2} g(x)$

(c) Find the values of $f(1)$ and $g(2)$.

(d) For the limits and function values above, which of these are you most confident in? What about the limit, function value, or graph of the function makes you confident about your answer?

Similarly, which of these are you the least confident in? What about the limit, function value, or graph of the function makes you not have confidence in your answer?

Activity 1.3.2 From Estimating to Evaluating Limits (Part 2). Let's consider the following graphs of functions $f(x)$ and $g(x)$, now with the added labels of the equations defining each part of these functions.

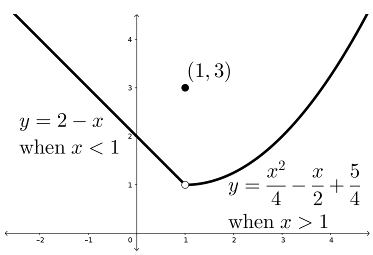


Figure 1.3.3 Graph of the function $f(x)$.

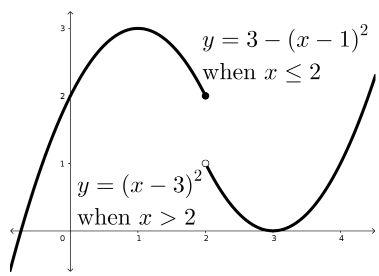


Figure 1.3.4 Graph of the function $g(x)$.

(a) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim_{x\to 1^-} f(x)$
- $\lim_{x\to 1^+} f(x)$
- $\lim_{x\to 1} f(x)$

(b) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim_{x\to 2^-} g(x)$
- $\lim_{x\to 2^+} g(x)$
- $\lim_{x\to 2} g(x)$

(c) Does the addition of the function rules change the level of confidence you have in these answers? What limits are you more confident in with this added information?

(d) Consider these functions without their graphs:

$$f(x) = \begin{cases} 2 - x & \text{when } x < 1 \\ 3 & \text{when } x = 1 \\ \frac{x^2}{4} - \frac{x}{2} + \frac{5}{4} & \text{when } x > 1 \end{cases}$$

$$g(x) = \begin{cases} 3 - (x-1)^2 & \text{when } x \leq 2 \\ (x-3)^2 & \text{when } x > 2 \end{cases}$$

Find the limits $\lim_{x \to 1} f(x)$ and $\lim_{x \to 2} g(x)$. Compare these values of $f(1)$ and $g(2)$: are they related at all?

1.3.2 Limit Properties

Theorem 1.3.5 Combinations of Limits. *If $f(x)$ and $g(x)$ are two functions defined at x-values around, but maybe not at, $x = a$ and $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist, then we can evaluate limits of combinations of these functions.*

1. Sums: *The limit of the sum of $f(x)$ and $g(x)$ is the sum of the limits of $f(x)$ and $g(x)$:*

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

2. Differences: *The limit of a difference of $f(x)$ and $g(x)$ is the difference of the limits of $f(x)$ and $g(x)$:*

$$\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

3. Coefficients: *If k is some real number coefficient, then:*

$$\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x)$$

4. Products: *The limit of a product of $f(x)$ and $g(x)$ is the product of the limits of $f(x)$ and $g(x)$:*

$$\lim_{x \to a} (f(x) \cdot g(x)) = \left(\lim_{x \to a} f(x)\right) \cdot \left(\lim_{x \to a} g(x)\right)$$

5. Quotients: *The limit of a quotient of $f(x)$ and $g(x)$ is the quotient of the limits of $f(x)$ and $g(x)$ (provided that you do not divide by 0):*

$$\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\left(\lim_{x \to a} f(x)\right)}{\left(\lim_{x \to a} g(x)\right)} \qquad \left(\textit{if } \lim_{x \to a} g(x) \neq 0\right)$$

Theorem 1.3.6 Limits of Two Basic Functions. *Let a be some real number.*

1. Limit of a Constant Function: *If k is some real number constant, then:*

$$\lim_{x \to a} k = k$$

2. Limit of the Identity Function:

$$\lim_{x \to a} x = a$$

Activity 1.3.3 Limits of Polynomial Functions. We're going to use a combination of properties from Theorem 1.3.5 and Theorem 1.3.6 to think a bit more deeply about polynomial functions. Let's consider a polynomial function:

$$f(x) = 2x^4 - 4x^3 + \frac{x}{2} - 5$$

(a) We're going to evaluate the limit $\lim_{x \to 1} f(x)$. First, use the properties from Theorem 1.3.5 to re-write this limit as 4 different limits added or subtracted together.

Answer.

$$\lim_{x \to 1}(2x^4) - \lim_{x \to 1}(4x^3) + \lim_{x \to 1}\left(\frac{x}{2}\right) - \lim_{x \to 1} 5$$

(b) Now, for each of these limits, re-write them as products of things until you have only limits of constants and identity functions, as in Theorem 1.3.6. Evaluate your limits.

Hint.

$$2\left(\lim_{x \to 1} x\right)^4 - 4\left(\lim_{x \to 1} x\right)^3 + \frac{1}{2}\left(\lim_{x \to 1} x\right) - \lim_{x \to 1} 5$$

(c) Based on the definition of a limit (Definition 1.2.1), we normally say that $\lim_{x \to 1} f(x)$ is not dependent on the value of $f(1)$. Why do we say this?

(d) Compare the values of $\lim_{x \to 1} f(x)$ and $f(1)$. Why do these values feel connected?

(e) Come up with a new polynomial function: some combination of coefficients with x's raised to natural number exponents. Call your new polynomial function $g(x)$. Evaluate $\lim_{x \to -1} g(x)$ and compare the value to $g(-1)$. Explain why these values are the same.

(f) Explain why, for any polynomial function $p(x)$, the limit $\lim_{x \to a} p(x)$ is the same value as $p(a)$.

Theorem 1.3.7 Limits of Polynomials. *If $p(x)$ is a polynomial function and a is some real number, then:*

$$\lim_{x \to a} p(x) = p(a)$$

1.4 First Indeterminate Forms

Activity 1.4.1 Limits of (Slightly) Different Functions.

(a) Using the graph of $f(x)$ below, approximate $\lim_{x \to 1} f(x)$.

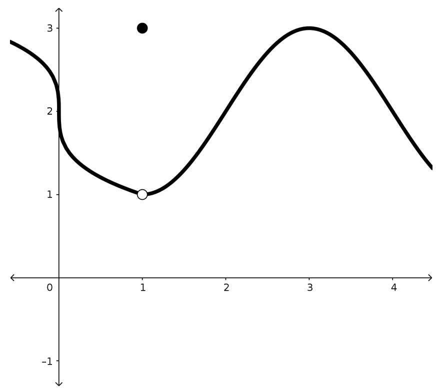


Figure 1.4.1

(b) Using the graph of the slightly different function $g(x)$ below, approximate $\lim_{x\to 1} g(x)$.

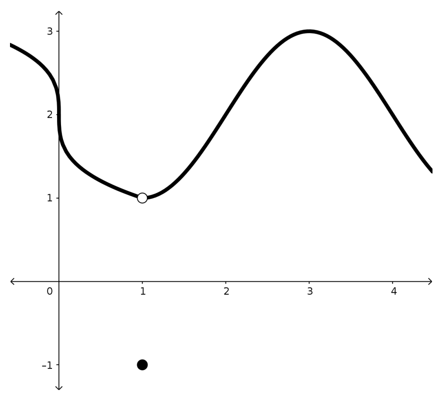


Figure 1.4.2

(c) Compare the values of $f(1)$ and $g(1)$ and discuss the impact that this difference had on the values of the limits.

(d) For the function $r(t)$ defined below, evaluate the limit $\lim_{x\to 4} r(t)$.

$$r(t) = \begin{cases} 2t - \frac{4}{t} & \text{when } t < 4 \\ 8 & \text{when } t = 4 \\ t^2 - t - 5 & \text{when } t > 4 \end{cases}$$

(e) For the slightly different function $s(t)$ defined below, evaluate the limit $\lim_{x\to 4} s(t)$.

$$s(t) = \begin{cases} 2t - \frac{4}{t} & \text{when } t \leq 4 \\ t^2 - t - 5 & \text{when } t > 4 \end{cases}$$

(f) Do the changes in the way that the function was defined impact the evaluation of the limit at all? Why not?

Theorem 1.4.3 Limits of (Slightly) Different Functions. *If $f(x)$ and $g(x)$ are two functions defined at x-values around a (but maybe not at $x = a$ itself) with $f(x) = g(x)$ for the x-values around a but with $f(a) \neq g(a)$ then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, if the limits exist.*

1.4.1 A First Introduction to Indeterminate Forms

Definition 1.4.4 Indeterminate Form. We say that a limit has an **indeterminate form** if the general structure of the limit could take on any different value, or not exist, depending on the specific circumstances.

For instance, if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then we say that the limit $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)$ has an indeterminate form. We typically denote this using the informal symbol $\frac{0}{0}$, as in:

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) \overset{?}{\to} \frac{0}{0}.$$

◊

Activity 1.4.2

(a) Were going to evaluate $\lim_{x\to 3}\left(\frac{x^2-7x+12}{x-3}\right)$.

- First, check that we get the indeterminate form $\frac{0}{0}$ when $x \to 3$.
- Now we want to find a new function that is equivalent to $f(x) = \frac{x^2-7x+12}{x-3}$ for all x-values other than $x = 3$. Try factoring the numerator, $x^2 - 7x + 12$. What do you notice?
- "Cancel" out any factors that show up in the numerator and denominator. Make a special note about what that factor is.
- This function is equivalent to $f(x) = \frac{x^2-7x+12}{x-3}$ except at $x = 3$. The difference is that this function has an actual function output at $x = 3$, while $f(x)$ doesn't. Evaluate the limit as $x \to 3$ for your new function.

(b) Now we'll evaluate a new limit: $\lim_{x\to 1}\left(\frac{\sqrt{x^2+3}-2}{x^2-5x+4}\right)$.

- First, check that we get the indeterminate form $\frac{0}{0}$ when $x \to 1$.
- Now we want a new function that is equivalent to $g(x) = \frac{\sqrt{x^2+3}-2}{x^2-5x+4}$ for all x-values other than $x = 1$. Try multiplying the numerator and the denominator by $(\sqrt{x^2+3}+2)$. We'll call this the "conjugate" of the numerator.
- In your multiplication, confirm that $(\sqrt{x^2+3}-2)(\sqrt{x^2+3}+2) = (x^2+3)-4$.
- Try to factor the new numerator and denominator. Do you notice anything? Can you "cancel" anything? Make another note of what factor(s) you cancel.
- This function is equivalent to $g(x) = \frac{\sqrt{x^2+3}-2}{x^2-5x+4}$ except at $x = 1$. The difference is that this function has an actual function output at $x = 1$, while $g(x)$ doesn't. Evaluate the limit as $x \to 1$ for your new function.

(c) Our last limit in this activity is going to be $\lim_{x\to -2}\left(\frac{3-\frac{3}{x+3}}{x^2+2x}\right)$.

- Again, check to see that we get the indeterminate form $\frac{0}{0}$ when $x \to -2$.
- Again, we want a new function that is equivalent to $h(x) = \dfrac{3 - \frac{3}{x+3}}{x^2 + 2x}$ for all x-values other than $x = -2$. Try completing the subtraction in the numerator, $3 - \frac{3}{x+3}$, using "common denominators."
- Try to factor the new numerator and denominator(s). Do you notice anything? Can you "cancel" anything? Make another note of what factor(s) you cancel.
- For the final time, we've found a function that is equivalent to $h(x) = \dfrac{3 - \frac{3}{x+3}}{x^2 + 2x}$ except at $x = -2$. The difference is that this function has an actual function output at $x = -2$, while $h(x)$ doesn't. Evaluate the limit as $x \to -2$ for your new function.

(d) In each of the previous limits, we ended up finding a factor that was shared in the numerator and denominator to cancel. Think back to each example and the factor you found. Why is it clear that these *must* have been the factors we found to cancel?

(e) Let's say we have some new function $f(x)$ where $\lim_{x\to 5} f(x) \stackrel{?}{\to} \dfrac{0}{0}$. You know, based on these examples, that you're going to apply *some* algebra trick to re-write your function, factor, and cancel. Can you predict what you will end up looking for to cancel in the numerator and denominator? Why?

1.4.2 What if There Is No Algebra Trick?

We've seen some nice examples above where we were able to use some algebra to manipulate functions in such was as to force some shared factor in the numerator and denominator into revealing itself. From there, we were able to apply Theorem 1.4.3 and swap out our problematic function with a new one, knowing that the limit would be the same.

But what if we can't do that? What if the specific structure of the function seems *resistent* somehow to our attempts at wielding algebra?

This happens a lot, and we'll investigate some more of those types of limits in Section **??**. For now, though, let's look at a very famous limit and reason our way through the indeterminate form.

Activity 1.4.3 Let's consider a new limit:

$$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}.$$

This one is strange!

(a) Notice that this function, $f(\theta) = \dfrac{\sin(\theta)}{\theta}$, is resistent to our algebra tricks:

- There's nothing to "factor" here, since our trigonometric function is not a polynomial.
- We can't use a trick like the "conjugate" to multiply and re-write, since there's no square roots and also only one term in the numerator.
- There aren't any fractions that we can combine by addition or subtraction.

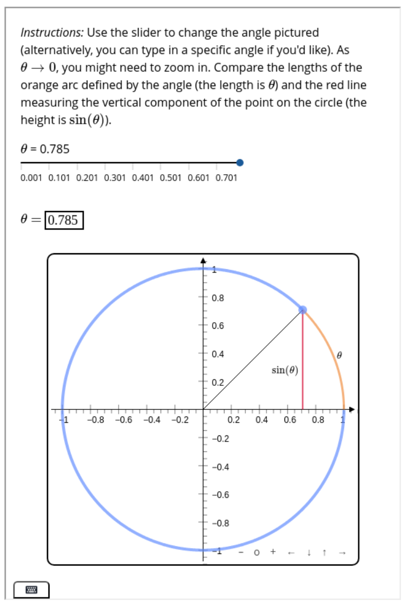
(b) Be frustrated at this new limit for resisting our algebra tricks.

(c) Now let's think about the meaning of $\sin(\theta)$ and even θ in general. In this text, we will often use Greek letters, like θ, to represent angles. In general, these angles will be measured in radians (unless otherwise specified). So what does the sine function *do* or *tell us*? What is a radian?

Hint 1. On the unit circle, if we plot some point at an angle of θ, then the coordinates of that point can be represented with trig functions! Which ones?

Hint 2. The length of the curve defining a unit circle is 2π. This also corresponds to the angle we would use to represent moving all the way around the circle. What must the length of the portion of the circle be up to some point at an angle θ?

(d) Let's visualize our limit, then, by comparing the length of the arc and the height of the point as $\theta \to 0$.



(e) Explain to yourself, until you are absolutely certain, why the two lengths *must* be the same in the limit as $\theta \to 0$. What does this mean about $\lim_{\theta \to 0} \dfrac{\sin(\theta)}{\theta}$?

1.5 Limits Involving Infinity

Two types of limits involving infinity. In both cases, we'll mostly just consider what happens when we divide by small things and what happens when we divide by big things. We can summarize this here, though:

Fractions with small denominators are big, and fractions with big denominators are small.

1.5.1 Infinite Limits

Activity 1.5.1 What Happens When We Divide by 0? First, let's make sure we're clear on one thing: there is no real number than is represented as some other number divided by 0.

When we talk about "dividing by 0" here (and in Section 1.4), we're talking about the behavior of some function in a limit. We want to consider what it might look like to have a function that involves division where the denominator *gets arbitrarily close to 0* (or, the limit of the denominator is 0).

(a) Remember when, once upon a time, you learned that dividing one a number by a fraction is the same as multiplying the first number by the reciprocal of the fraction? Why is this true?

(b) What is the relationship between a number and its reciprocal? How does the size of a number impact the size of the reciprocal? Why?

(c) Consider $12 \div N$. What is the value of this division problem when:

- $N = 6$?
- $N = 4$?
- $N = 3$?
- $N = 2$?
- $N = 1$?

(d) Let's again consider $12 \div N$. What is the value of this division problem when:

- $N = \frac{1}{2}$?
- $N = \frac{1}{3}$?
- $N = \frac{1}{4}$?
- $N = \frac{1}{6}$?
- $N = \frac{1}{1000}$?

(e) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to 0^+$? Note that this means that the x-values we're considering most are very small and positive.

(f) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to 0^-$? Note that this means that the x-values we're considering most are very small and negative.

Definition 1.5.1 Infinite Limit. We say that a function $f(x)$ has an **infinite limit** at a if $f(x)$ is arbitrarily large (positive or negative) when x is sufficiently close to, but not equal to, $x = a$.

We would then say, depending on the sign of the values of $f(x)$, that:

$$\lim_{x \to a^-} f(x) = \pm\infty \qquad\qquad \lim_{x \to a^+} f(x) = \pm\infty.$$

If the sign of both one-sided limits are the same, we can say that $\lim\limits_{x \to a} f(x) = \pm\infty$ (depending on the sign), but it is helpful to note that, by the definition of the Limit of a Function, this limit does not exist, since $f(x)$ is not arbitrarily close to a single real number. ◊

Theorem 1.5.2 Dividing by 0 in a Limit. *If* $f(x) = \dfrac{g(x)}{h(x)}$ *with* $\lim_{x\to a} g(x) \neq 0$ *and* $\lim_{x\to a} h(x) = 0$*, then* $f(x)$ *has an Infinite Limit at* a*. We will often denote this behavior as:*

$$\lim_{x\to a} f(x) \overset{?}{\to} \frac{\#}{0}$$

where # is meant to be some shorthand representation of a non-zero limit in the numerator (often, but not necessarily, some real number).

Evaluating Infinite Limits.

Once we know that $\lim_{x\to a} f(x) \overset{?}{\to} \frac{\#}{0}$, we know a bunch of information right away!

- This limit doesn't exist.
- The function $f(x)$ has a vertical asymptote at $x = a$, causing these unbounded y-values near $x = a$.
- The one sided limits *must* be either ∞ or $-\infty$.
- We only need to focus on the sign of the one sided limits! And signs of products and quotients are easy to follow.

So a pretty typical process is to factor as much as we can, and check the sign of each factor (in a numerator or denominator) as $x \to a^-$ and $x \to a^+$. From there, we can find the sign of $f(x)$ in both of those cases, which will tell us the one-sided limit.

Example 1.5.3 For each function, find the relevant one-sided limits at the input-value mentioned. If you can use a two-sided limit statement to discuss the behavior of the function around this input-value, then do so.

(a) $\left(\dfrac{2x^2 - 5x + 1}{x^2 + 8x + 16}\right)$ and $x = -4$

(b) $\left(\dfrac{4x^2 - x^5}{x^2 - 4x + 3}\right)$ and $x = 1$

(c) $\sec(\theta)$ and $\theta = \dfrac{\pi}{2}$

□

1.5.2 End Behavior Limits

Activity 1.5.2 What Happens When We Divide by Infinity? Again, we need to start by making something clear: if we were really going to try divide some real number by infinity, then we would need to re-build our definition of what it means to divide. In the context we're in right now, we only have division defined as an operation for real (and maybe complex) numbers. Since infinity is neither, then we will not literally divide by infinity.

When we talk about ”dividing by infinity” here, we're again talking about the behavior of some function in a limit. We want to consider what it might look like to have a function that involves division where the denominator *gets arbitrarily large (positive or negative)* (or, the limit of the denominator is infinite).

(a) Let's again consider $12 \div N$. What is the value of this division problem when:

- $N = 1$?
- $N = 6$?
- $N = 12$?
- $N = 24$?
- $N = 1000$?

(b) Let's again consider $12 \div N$. What is the value of this division problem when:

- $N = -1$?
- $N = -6$?
- $N = -12$?
- $N = -24$?
- $N = -1000$?

(c) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to \infty$? Note that this means that the x-values we're considering most are very large and positive.

(d) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to -\infty$? Note that this means that the x-values we're considering most are very large and negative.

(e) Why is there no difference in the behavior of $f(x)$ as $x \to \infty$ compared to $x \to -\infty$ when the sign of the function outputs are opposite ($f(x) > 0$ when $x \to \infty$ and $f(x) < 0$ when $x \to -\infty$)?

Definition 1.5.4 Limit at Infinity. If $f(x)$ is defined for all large and positive x-values and $f(x)$ gets arbitrarily close to the single real number L when x gets sufficiently large, then we say:

$$\lim_{x\to\infty} f(x) = L.$$

Similarly, if $f(x)$ is defined for all large and negative x-values and $f(x)$ gets arbitrarily close to the single real number L when x gets sufficiently negative, then we say:

$$\lim_{x\to-\infty} f(x) = L.$$

In the case that $f(x)$ has a **limit at infinity** that exists, then we say $f(x)$ has a horizontal asymptote at $y = L$.

Lastly, if $f(x)$ is defined for all large and positive (or negative) x-values and $f(x)$ gets arbitrarily large and positive (or negative) when x gets sufficiently large (or negative), then we could say:

$$\lim_{x\to-\infty} f(x) = \pm\infty \text{ or } \lim_{x\to\infty} f(x) = \pm\infty.$$

◊

Because the primary focus for limits at infinity is the end behavior of a function, we will often refer to these limits as **end behavior limits**.

Theorem 1.5.5 End Behavior of Reciprocal Power Functions. *If p is a positive real number, then:*

$$\lim_{x\to\infty}\left(\frac{1}{x^p}\right)=0 \text{ and } \lim_{x\to-\infty}\left(\frac{1}{x^p}\right)=0.$$

Theorem 1.5.6 Polynomial End Behavior Limits. *For some polynomial function:*

$$p(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_2x^2+a_1x+a_0$$

with n a positive integer (the degree) and all of the coefficients $a_0, a_1, ..., a_n$ real numbers (with $a_n \neq 0$), then

$$\lim_{x\to\infty}p(x)=\lim_{x\to\infty}a_nx^n$$

That is, the leading term (the term with the highest exponent) defines the end behavior for the whole polynomial function.

Proof. Consider the polynomial function:

$$p(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_2x^2+a_1x+a_0$$

where n is some integer and a_k is a real number for $k=0,1,2,...,n$. For simplicity, we will consider only the limit as $x\to\infty$, but we could easily repeat this exact proof for the case where $x\to-\infty$.

Before we consider this limit, we can factor out x^n, the variable with the highest exponent:

$$\begin{aligned}p(x)&=a_nx^n+a_{n-1}x^{n-1}+...+a_2x^2+a_1x+a_0\\&=x^n\left(\frac{a^nx^n}{x^n}+\frac{a_{n-1}x^{n-1}}{x^n}+...+\frac{a_2x^2}{x^n}+\frac{a_1x}{x^n}+\frac{a_0}{x^n}\right)\\&=x^n\left(a_n+\frac{a_{n-1}}{x}+...+\frac{a_2}{x^{n-2}}+\frac{a_1}{x^{n-1}}+\frac{a_0}{x^n}\right)\end{aligned}$$

Now consider the limit of this product:

$$\begin{aligned}\lim_{x\to\infty}p(x)&=\lim_{x\to\infty}x^n\left(a_n+\frac{a_{n-1}}{x}+...+\frac{a_2}{x^{n-2}}+\frac{a_1}{x^{n-1}}+\frac{a_0}{x^n}\right)\\&=\left(\lim_{x\to\infty}x^n\right)\left(\lim_{x\to\infty}a_n+\frac{a_{n-1}}{x}+...+\frac{a_2}{x^{n-2}}+\frac{a_1}{x^{n-1}}+\frac{a_0}{x^n}\right)\end{aligned}$$

We can see that in the second limit, we have a single constant term, a_n, followed by reciprocal power functions. Then, due to Theorem 1.5.5, we know that the second limit will by a_n, since the reciprocal power functions will all approach 0.

$$\begin{aligned}\lim_{x\to\infty}p(x)&=\left(\lim_{x\to\infty}x^n\right)\left(\lim_{x\to\infty}a_n+\frac{a_{n-1}}{x}+...+\frac{a_2}{x^{n-2}}+\frac{a_1}{x^{n-1}}+\frac{a_0}{x^n}\right)\\&=\left(\lim_{x\to\infty}x^n\right)(a_n+0+...+0+0+0)\\&=\left(\lim_{x\to\infty}x^n\right)(a_n)\\&=\lim_{x\to\infty}a_nx^n\end{aligned}$$

And so $\lim_{x\to\infty}p(x)=\lim_{x\to\infty}a_nx^n$ as we claimed. ■

Example 1.5.7 For each function, find the limits as $x\to\infty$ and $x\to-\infty$.

(a) $\left(\dfrac{2x^2 - 5x + 1}{x^2 + 8x + 16}\right)$

(b) $\left(\dfrac{4x^2 - x^5}{x^2 - 4x + 3}\right)$

(c) $\dfrac{|x|}{3x}$

□

Activity 1.5.3 Matching the Limits.

(a) We're going to look at four graphs of functions, as well as a list of limit statements. Match the limit statements with the graphs that match that behavior. Note that is is possible for a limit to be relevant on more than one graph.

(b) Now consider these four function definitions. Using your knowledge of limits, as well as the matching you've already done, match the definitions of these four functions with the graphs that go with them, and then also the limits that are relevant. (These limits will already be matched with the graphs, so you don't need to do further work here).

1.6 The Squeeze Theorem

Activity 1.6.1 A Weird End Behavior Limit. In this activity, we're going to find the following limit:

$$\lim_{x\to\infty}\left(\frac{\sin^2(x)}{x^2+1}\right).$$

This limit is a bit weird, in that we really haven't looked at trigonometric functions that much. We're going to start by looking at a different limit in the hopes that we can eventually build towards this one.

(a) Consider, instead, the following limit:

$$\lim_{x\to\infty}\left(\frac{1}{x^2+1}\right).$$

Find the limit and connect the process or intuition behind it to at least one of the results from this text.

Hint 1. Start with Theorem 1.3.5 to think about the numerator and denominator separately.

Hint 2. Can you use Theorem 1.5.6 in the denominator?

Hint 3. Is Theorem 1.5.5 relevant?

(b) Let's put this limit aside and briefly talk about the sine function. What are some things to remember about this function? What should we know? How does it behave?

(c) What kinds of values doe we expect $\sin(x)$ to take on for different values of x?

$$\underline{\qquad\qquad} \le \sin(x) \le \underline{\qquad\qquad}$$

(d) What happens when we square the sine function? What kinds of values can that take on?

$$\underline{\qquad\qquad} \leq \sin^2(x) \leq \underline{\qquad\qquad}$$

(e) Think back to our original goal: we wanted to know the end behavior of $\dfrac{\sin^2(x)}{x^2+1}$. Right now we have two bits of information:

- We know $\lim\limits_{x\to\infty} \left(\dfrac{1}{x^2+1}\right)$.
- We know some information about the behavior of $\sin^2(x)$. Specifically, we have some bounds on its values.

Can we combine this information?

In your inequality above, multiply $\left(\frac{1}{x^2+1}\right)$ onto all three pieces of the inequality. Make sure you're convinced about the direction or order of the inequality and whether or not it changes with this multiplication.

$$\underbrace{\frac{\qquad\qquad}{x^2+1}}_{\text{call this } f(x)} \leq \frac{\sin^2(x)}{x^2+1} \leq \underbrace{\frac{\qquad\qquad}{x^2+1}}_{\text{call this } h(x)}$$

(f) For your functions $f(x)$ and $h(x)$, evaluate $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to\infty} h(x)$.

(g) What do you think this means about the limit we're interested in, $\lim\limits_{x\to\infty} \left(\dfrac{\sin^2(x)}{x^2+1}\right)$?

Theorem 1.6.1 The Squeeze Theorem. *For some functions $f(x)$, $g(x)$, and $h(x)$ which are all defined and ordered $f(x) \leq g(x) \leq h(x)$ for x-values near $x = a$ (but not necessarily at $x = a$ itself), and for some real number L, if we know that*

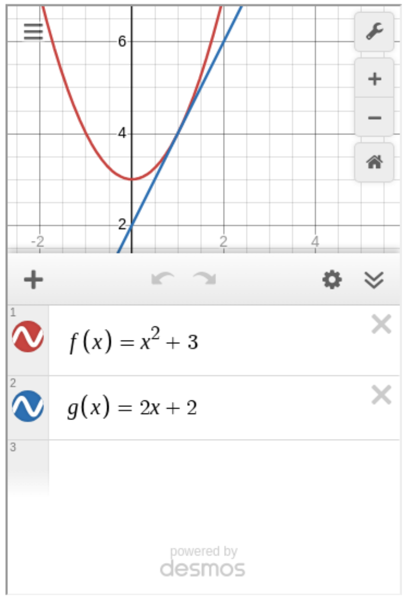
$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$$

then we also know that $\lim\limits_{x\to a} g(x) = L$.

Activity 1.6.2 Sketch This Function Around This Point.

(a) Sketch or visualize the functions $f(x) = x^2 + 3$ and $h(x) = 2x + 2$, especially around $x = 1$.

Hint.



(b) Sketch some other function, $g(x)$ where $f(x) \leq g(x) \leq h(x)$ for the x-values around (byt maybe not at) $x = 1$.

(c) Use the Squeeze Theorem to evaluate and explain $\lim_{x \to 1} g(x)$ for your function $g(x)$.

(d) Is this limit dependent on the specific version of $g(x)$ that you sketched? Would this limit be different for someone else's choice of $g(x)$ given the same parameters?

(e) What information must be true (if anything) about $\lim_{x \to 3} g(x)$ and $\lim_{x \to 0} g(x)$?
Do we know that these limits exist? If they do, do we have information about their values?

1.7 Continuity and the Intermediate Value Theorem

1.7.1 Continuity as Connectedness

1.7.2 Continuity as Classification

Definition 1.7.1 Continuous at a Point. The function $f(x)$ is **continuous** at an x-value in the domain of $f(x)$ if $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

If $f(x)$ is not continuous at $x = a$, but one of the one-sided limits is equal to the function output, then we can define **directional continuity** at that point:

- We say $f(x)$ is **continuous on the left** at $x = a$ when $\lim_{x \to a^-} f(x) = f(a)$.
- We say $f(x)$ is **continuous on the right** at $x = a$ when $\lim_{x \to a^+} f(x) = f(a)$.

◊

Definition 1.7.2 Continuous on an Interval. We say that $f(x)$ is **continuous on the interval** (a,b) if $f(x)$ is continuous at every x-value with $a<x<b$.

If $f(x)$ is continuous on the right at $x=a$ and/or continuous on the left at $x=b$, then we will say that $f(x)$ is continuous on the interval $[a,b)$, $(a,b]$, or $[a,b]$, whichever is relevant. ♢

1.7.3 Discontinuities

Where is a Function not Continuous?

Most of the functions that we consider in this text will be continuous everywhere that it makes sense: on their domain. That is, if there is a point defined at some x-value, it is likely that the function's limit matches the y-value of the point. More specifically, though:

- A function is discontinuous at any location that results in an infinite limit. These are locations where $f(x)$ is undefined and the limit is infinite (and so doesn't exist).
- A function is, in general, discontinuous wherever it is undefined. This seems silly to say! We probably could have left this unsaid.
- A function that is defined as a piecewise function could be discontinuous at locations where the pieces meet: maybe the limit doesn't exist, or maybe the function value is not defined, or maybe the limit exists and the function value is defined but they do not match.

1.7.4 Intermediate Value Theorem

Theorem 1.7.3 Intermediate Value Theorem. *If $f(x)$ is a function that is continuous on $[a,b]$ with $f(a)\neq f(b)$ and L is any real number between $f(a)$ and $f(b)$ (either $f(a)<L<f(b)$ or $f(b)<L<f(a)$), then there exists some c between a and b ($a<c<b$) such that $f(c)=L$.*

This theorem was stated as early as the 5th century BCE by Bryson of Heraclea. Back then, a really interesting problem was related to "squaring the circle." That is, given a circle with some measurable radius, can we construct a square with equal area? This is obviously true, in that we can just use a square with the side length $r\sqrt{\pi}$. What we typically mean by "construct," though, is to create this square using only a compass and straightedge (a ruler without length markings) and only a finite number of steps. This was finally proven to be impossible in 1882, approximately 2300 years later.

Bryson of Heraclea knew that the square itself existed (even if he couldn't construct it) because he was able to find a circle with area less than the square (by inscribing a circle inside of the square) and a circle with area greater than the square (where the square is inscribed in the circle). Since he posited that he could increase the size of the circle in a continuous manner (without using those words), he claimed that a square with area equal to that of the circle must exist, since the area of the circle passes through all values from the smaller area to the larger area.

Chapter 2

Derivatives

2.1 Introduction to Derivatives

We'll start this off by thinking about slopes. Before we begin, you should be able to answer the following questions:

- What *is* a slope? How could you describe it?
- How do you calculate the slope of a line between two points?
- If we have a function $f(x)$ and we pick two points on the curve of the function, what does the slope of a straight line connecting the two points tell us? What kind of behavior about $f(x)$ does this slope describe?

2.1.1 Defining the Derivative

Activity 2.1.1 Thinking about Slopes. We're going to calculate and make some conjectures about slopes of lines between points, where the points are on the graph of a function. Let's define the following function:

$$f(x) = \frac{1}{x+2}.$$

(a) We're going to calculate a lot of slopes! Calculate the slope of the line connecting each pair of points on the curve of $f(x)$:

- $(-1, f(-1))$ and $(0, f(0))$
- $(-0.5, f(-0.5))$ and $(0, f(0))$
- $(-0.1, f(-0.1))$ and $(0, f(0))$
- $(-0.001, f(-.001))$ and $(0, f(0))$

(b) Let's calculate another group of slopes. Find the slope of the lines connecting these pairs of points:

- $(0, f(0))$ and $(1, f(1))$
- $(0, f(0))$ and $(0.5, f(0.5))$
- $(0, f(0))$ and $(0.1, f(0.1))$
- $(0, f(0))$ and $(0.001, f(0.001))$

(c) Just to make it clear what we've done, lay out your slopes in this table:

Between $(0, f(0))$ **and...**	**Slope**
$(1, f(1))$	
$(0.5, f(0.5))$	
$(0.1, f(0.1))$	
$(0.01, f(0.01))$	
$(-0.01, f(-0.01))$	
$(-0.1, f(-0.1))$	
$(-0.5, f(-0.5))$	
$(-1, f(-1))$	

(d) Now imagine a line that is tangent to the graph of $f(x)$ at $x = 0$. We are thinking of a line that touches the graph at $x = 0$, but runs along side of the curve there instead of through it.

Make a conjecture about the slope of this line, using what we've seen above.

(e) Can you represent the slope you're thinking of above with a limit? What limit are we approximating in the slope calculations above? Set up the limit and evaluate it, confirming your conjecture.

Activity 2.1.2 Finding a Tangent Line. Let's think about a new function, $g(x) = \sqrt{2 - x}$. We're going to think about this function around the point at $x = 1$.

(a) Ok, we are going to think about this function at this point, so let's find the coordinates of the point first. What's the y-value on our curve at $x = 1$?

(b) Use a limit similar to the one you constructed in Activity 2.1.1 to find the slope of the line tangent to the graph of $g(x)$ at $x = 1$.

(c) Now that you have a slope of this line, and the coordinates of a point that the line passes through, can you find the equation of the line?

Definition 2.1.1 Derivative at a Point. For a function $f(x)$, we say that the **derivative** of $f(x)$ at $x = a$ is:

$$f'(a) = \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \right)$$

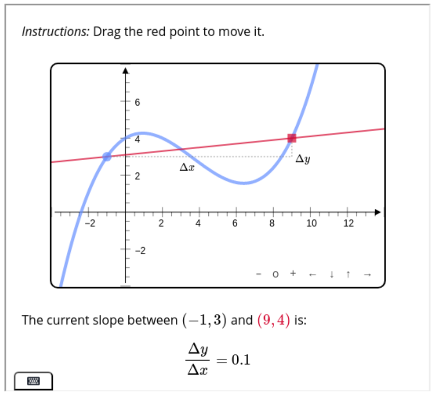
provided that the limit exists.

If $f'(a)$ exists, then we say $f(x)$ is **differentiable** at a. ◊

We can investigate this definition visually. Consider the function $f(x)$ plotted below, where we will look at the point $(-1, f(-1))$. In the definition of the limit, we'll let $a = -1$, and so consider:

$$\lim_{x \to -1} \left(\frac{f(x) - f(-1)}{x - (-1)} \right).$$

Can you estimate the limit of the slope of the tangent line as $x \to -1$?



Does it look like the limit of the slope between $(-1, f(-1))$ and $(x, f(x))$ exists as $x \to -1$? What do you think it is?

2.1.2 Calculating a Bunch of Slopes at Once

Activity 2.1.3 Calculating a Bunch of Slopes. Let's do this all again, but this time we'll calculate the slope at a bunch of different points on the same function.

Let's use $j(x) = x^2 - 4$.

(a) Start calculating the following derivatives, using the definition of the Derivative at a Point:

- $j'(-2)$
- $j'(0)$
- $j'(1/3)$
- $j'(-1)$

(b) Stop calculating the above derivatives when you get tired/bored of it. How many did you get through?

(c) Notice how repetitive this is: on one hand, we have to set up a completely different limit each time (since we're looking at a different point on the function each time). On the other hand, you might have noticed that the work is all the same: you factor and cancel over and over. These limits are all ones that we covered in Section 1.4 First Indeterminate Forms, and so it's no surprise that we keep using the same algebra manipulations over and over again to evaluate these limits.

Do you notice any patterns, any connections between the x-value you used for each point and the slope you calculated at that point? You might need to go back and do some more.

(d) Try to evaluate this limit in general:

$$\begin{aligned} j'(a) &= \lim_{x \to a} \left(\frac{j(x) - j(a)}{x - a} \right) \\ &= \lim_{x \to a} \left(\frac{(x^2 - 4) - (a^2 - 4)}{x - a} \right). \end{aligned}$$

Remember, you know how this goes! You're going to do the same sorts of algebra that you did earlier!

What is the formula, the pattern, the way of finding the slope on the $j(x)$ function at any x-value, $x = a$?

(e) Confirm this by using your new formula to re-calculate the following derivatives:

- $j'(-2)$
- $j'(0)$
- $j'(1/3)$
- $j'(-1)$

We're going to try to think about the derivative as something that can be calculated in general, as well as something that can be calculated at a point. We'll define a new way of calculating it, still a limit of slopes, that will be a bit more general.

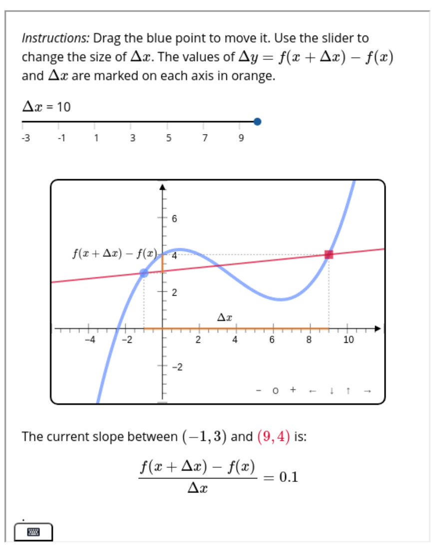
Definition 2.1.2 The Derivative Function. For a function $f(x)$, the derivative of $f(x)$, denoted $f'(x)$, is:

$$f'(x) = \lim_{\Delta x \to 0} \left(\frac{f(x + \Delta x) - f(x)}{\Delta x} \right)$$

for x-values in the domain of $f(x)$ where this limit exists. ◊

This definition feels pretty different, but we can hopefully notice that this is really just calculating a slope. Notice, in the following plot, that there is a significant difference. In the visualization of the Derivative at a Point, the first point was fixed into place and the second point was the one that we moved and changed. It was the one with the variable x-value.

Notice in the following visualization that the *first* point is the one that is moveable while the *second* point is defined based on the first one (and the horizontal difference between the points, Δx). This means that we don't need to define one specific point, and can find the slope of the line tangent to $f(x)$ at some changing x-value.



2.2 Interpreting Derivatives

What is a derivative?

This can feel like a silly question, since we're calculating it and getting used to finding them. But what is it?

In this section, we just want to remind ourselves of what this object is, how we should hold it in our minds as we move through the course, and then practice being flexible with this interpretation.

2.2.1 The Derivative is a Slope

Activity 2.2.1 Interpreting the Derivative as a Slope. In Activity 2.1.1 Thinking about Slopes and Activity 2.1.2 Finding a Tangent Line, we built the idea of a derivative by calculating slopes and using them. Let's continue this by considering the function $f(x) = \dfrac{1}{x^2}$.

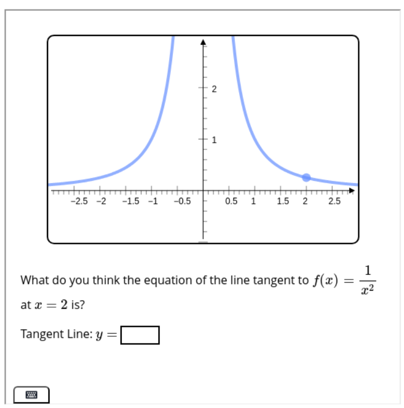
(a) Use Definition 2.1.1 Derivative at a Point to find $f'(2)$. What does this value represent?

(b) We want to plot the line that would be tangent to the graph of $f(x)$ at $x = 2$.

Remember that we can write the equation of a line in two ways:

- The equation of a line with slope m that passes through the point $(a, f(a))$ is:
$$y = m(x - a) + f(a).$$
- The equation of a line with slope m that passes the point $(0, b)$ (this is another way of saying that the y-intercept of the line is b) is:
$$y = mx + b.$$

Find the equation of the line tangent to $f(x)$ at $x = 2$. Add it to the graph of $f(x) = \dfrac{1}{x^2}$ below to check your equation.



(c) This tangent line is very similar to the actual curve of the function $f(x)$ near $x = 2$. Another way of saying this is that while the slope of $f(x)$ is not always the value you found for $f'(2)$, it is close to that for x-values near 2.

Use this idea of slope to predict where the y-value of our function will be at 2.01.

Hint. We know that slope is $\dfrac{\Delta y}{\Delta x}$ and we're using the fact that $\dfrac{dy}{dx} \approx \dfrac{\Delta y}{\Delta x}$ for small values of Δx.

Here, we have $\Delta x = 0.01$, so can you use the slope to approximate the corresponding Δy and figure out the new y-value?

(d) Compare this value with $f(2.01) = \dfrac{1}{2.01^2}$. How close was it?

2.2.2 The Derivative is a Rate of Change

Activity 2.2.2 Interpreting the Derivative as a Rate of Change. This is going to somewhat feel redundant, since maybe we know that a slope is really just a rate of change. But hopefully we'll be able to explore this a bit more and see how we can use a derivative to tell us information about some specific context.

Let's say that we want to model the speed of a car as it races along a strip of the road. By the time we start measuring it (we'll call this time 0), the position the car (along the straight strip of road) is:

$$s(t) = 73t + t^2,$$

where t is time measured in seconds and $s(t)$ is the position measured in feet. Let's say that this function is only relevant on the domain $0 \leq t \leq 15$. That is, we only model the position of the car for a 15-second window as it speeds past us.

(a) How far does the car travel in the 15 seconds that we model it? What was the car's average velocity on those 15 seconds?

(b) Calculate $s'(t)$, the derivative of $s(t)$, using Definition 2.1.2 The Derivative Function. What information does this tell us about our vehicle?

Hint. What is the rate at which the position (in feet) of the vehicle changes per unit time (in seconds)? What do we call this, and what are the units?

(c) Calculate $s'(0)$. Why is this smaller than the average velocity you found? What does that mean about the velocity of the car?

(d) If we call $v(t) = s'(t)$, then calculate $v'(t)$. Note that this is a derivative of a derivative.

(e) Find $v'(0)$. Why does this make sense when we think about the difference between the average velocity on the time interval and the value of $v(0)$ that we calculated?

(f) What does it mean when we notice that $v'(t)$ is constant? Explain this by interpreting it in terms of both the velocity of the vehicle as well as the position.

2.2.3 The Derivative is a Limit

Look back at the definition of Derivative at a Point. The end of it is interesting: "provided that the limit exists." We need to keep in mind that this is a limit, and so a derivative exists or fails to exist whenever that limit exists or fails to exist.

What are some ways that a limit fails to exist?

- A limit doesn't exist if the left-side limit and the right-side limit do not match: Theorem 1.2.4 Mismatched Limits.
- A limit doesn't exist if it is an Infinite Limit.

What do each of these situations look like when we're considering the limit of slopes?

When Does a Derivative Not Exist?

1. A derivative doesn't exist at points where the slopes on either side of the point don't match.
2. A derivative doesn't exist at points with vertical tangent lines.
3. A derivative doesn't exist at points where the function is not continuous.

2.2.4 The Derivative is a Function

Activity 2.2.3 Interpreting the Derivative as a Function. In Activity 2.1.3 Calculating a Bunch of Slopes, we calculated the derivative function for $j(x) = x^2 - 4$. Using the definition of The Derivative Function, we can see that $j'(x) = 2x$. Let's explore that a bit more.

(a) Sketch the graphs of $j(x) = x^2 - 4$ and $j'(x) = 2x$. Describe the shapes of these graphs.

(b) Find the coordinates of the point at $x = \frac{1}{2}$ on both the graph of $j(x)$ and $j'(x)$. Plot the point on each graph.

(c) Think back to our previous interpretations of the derivative: how do we interpret the y-value output you found for the j' function?

(d) Find the coordinates of another point at some other x-value on both the graph of $j(x)$ and $j'(x)$. Plot the point on each graph, and explain what the output of j' tells us at this point.

(e) Use your graph of $j'(x)$ to find the x-intercept of $j'(x)$. Locate the point on $j(x)$ with this same x-value. How do we know, visually, that this point is the x-intercept of $j'(x)$?

(f) Use your graph of $j'(x)$ to find where $j'(x)$ is positive. Pick two x-values where $j'(x) > 0$. What do you expect this to look like on the graph of $j(x)$? Find the matching points (at the same x-values) on the graph of $j(x)$ to confirm.

(g) Use your graph of $j'(x)$ to find where $j'(x)$ is negative. Pick two x-values where $j'(x) < 0$. What do you expect this to look like on the graph of $j(x)$? Find the matching points (at the same x-values) on the graph of $j(x)$ to confirm.

(h) Let's wrap this up with one final pair of points. Let's think about the point $(-3, 5)$ on the graph of $j(x)$ and the point $(-3, -6)$ on the graph of $j'(x)$. First, explain what the value of -6 (the output of j' at $x = -3$) means about the point $(-3, 5)$ on $j(x)$. Finally, why can we not use the value 5 (the output of j at $x = -3$) means about the point $(-3, -6)$ on $j'(x)$?

2.2.5 Notation for Derivatives

So far we've been using the "prime" notation to represent derivatives: the derivative of $f(x)$ is $f'(x)$. We will continue to use this notation, but we'll introduce a bunch of other ways of writing notation to represent the derivative.

Each new notation will emphasize some aspect of the derivative that will serve to be useful, even though they all represent essentially the same thing.

Function	**Derivative**	**Derivative at $x = a$**	**Emphasis**
$f(x)$	$f'(x)$	$f'(a)$	The derivative is a function. The function takes in x-value inputs and returns the slope of f at that x-value.
y	y'	$y'\Big\rvert_{x=a}$	We can find slopes on any curve, not just functions. This is sometimes also used as a way to simplify the notation, especially when we want to manipulate equations involving y'.
y	$\frac{dy}{dx}$	$\frac{dy}{dx}\Big\rvert_{x=a}$	The derivative is a slope. It is $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$, and we use dx and dy (called **differentials**) to represent Δx and Δy as the limits as $\Delta x \to 0$. This notation is also useful to tell us what the rate of change is: what is changing (in this case y) and what is it changing based on (in this case x).
$f(x)$	$\frac{d}{dx}(f(x))$	$\frac{d}{dx}(f(x))\Big\rvert_{x=a}$	The derivative is an action that we do to some function. We can call it an **operator**, although we won't formally define that term in this text. We'll look at this idea more in Section **??**. We can specify what we are expecting the input variable to be, based on the differential dx in the denominator.

2.3 Some Early Derivative Rules

We are going to break this topic into two parts:

1. We will try to find some common patterns or connections between derivatives and specific functions. For instance, when we use Definition 2.1.2 The Derivative Function to build a derivative, are there patterns in the work of evaluating that limit that will allow us to get through the limit work quickly? Can we group some functions together based on how we might deal with the limit?

2. We will try to think about derivatives a bit more generally and show how we can build some basic properties to help us think about differentiating variations of the functions that we recognize.

2.3.1 Derivatives of Common Functions

Activity 2.3.1 Derivatives of Power Functions. We're going to do a bit of pattern recognition here, which means that we will need to differentiate several different power functions. For our reference, a power function (in general) is a function in the form $f(x) = a(x^n)$ where n and a are real numbers, and $a \neq 0$.

Let's begin our focus on the power functions x^2, x^3, and x^4. We're going to use Definition 2.1.2 The Derivative Function a lot, so feel free to review it before we begin.

(a) Find $\frac{d}{dx}\left(x^2\right)$. As a brief follow up, compare this to the derivative $j'(x)$ that you found in Activity 2.1.3 Calculating a Bunch of Slopes. Why are they the same? What does the difference, the -4, in the $j(x)$ function do to the graph of it (compared to the graph of x^2) and why does this not impact the derivative?

Hint. Remember that the graph of $x^2 - 4$ has the same shape as the graph of x^2, but has been shifted down by 4 units. Why does this vertical shift not change the value of the derivative at any x-value?

(b) Find $\frac{d}{dx}\left(x^3\right)$.

Hint. Remember that $(x + \Delta x)^3 = (x + \Delta x)(x + \Delta x)(x + \Delta x)$

(c) Find $\frac{d}{dx}\left(x^4\right)$.

Hint. Remember that $(x+\Delta x)^4 = (x+\Delta x)(x+\Delta x)(x+\Delta x)(x+\Delta x)$

(d) Notice that in these derivative calculations, the main work is in multiplying $(x+\Delta x)^n$. Look back at the work done in all three of these derivative calculations and find some unifying steps to describe how you evaluate the limit/calculate the derivative *after* this tedious multiplication was finished. What steps did you do? Is it always the same thing?

Another way of stating this is: if I told you that I knew what $(x + \Delta x)^5$ was, could you give me some details on how the derivative limit would be finished?

(e) Finish the following derivative calculation:

$$\begin{aligned}
\frac{d}{dx}\left(x^5\right) &= \lim_{\Delta x \to 0}\left(\frac{(x + \Delta x)^5 - x^5}{\Delta x}\right) \\
&= \lim_{\Delta x \to 0}\left(\frac{(x^5 + 5x^4\Delta x + 10x^3\Delta x^2 + 10x^2\Delta x^3 + 5x\Delta x^4 + \Delta x^5) - x^5}{\Delta x}\right) \\
&=\rightsquigarrow \ldots
\end{aligned}$$

(f) Make a conjecture about the derivative of a power function in general, $\frac{d}{dx}(x^n)$.

Something to notice here is that the calculation in this limit is really dependent on knowing what $(x+\Delta x)^n$ is. When n is an integer with $n \geq 2$, this really just translates to multiplication. If we can figure out how to multiply $(x + \Delta x)^n$ in general, then this limit calculation will be pretty easy to do. We noticed that:

1. The first term of that multiplication will combine with the subtraction of x^n in the numerator and subtract to 0.

2. The rest of the terms in the multiplication have at least one copy of Δx, and so we can factor out Δx and "cancel" it with the Δx in the denominator.

3. Once this has done, we've escaped the portion of the limit that was giving us the $\frac{0}{0}$ indeterminate form, and so we can evaluate the limit as $\Delta x \to 0$. The result is just that whatever terms still have at least one remaining copy of Δx in it "go to" 0, and we're left with just the terms that do not have any copies of Δx in them.

Triangle binomial theorem for coefficients.

Theorem 2.3.1 Power Rule for Derivatives.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

We have shown that this is true for $n = 2, 3, 4, ...$, but this is also true for *any* value of n (including $n = 1$, non-integers, and non-positives). We will prove this more formally later (in Section **??**), and until then we will be free to use this result.

Example 2.3.2 Let's confirm this Power Rule for two examples that we are familiar with.

(a) Find the derivative $\frac{d}{dx}(\sqrt{x})$ using the limit definition of the derivative function. Note that $\sqrt{x} = x^{1/2}$ and $\frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$.

(b) Find the derivative $\frac{d}{dx}\left(\frac{1}{x}\right)$ using the limit definition of the derivative function. Note that $\frac{1}{x} = x^{-1}$ and $-\frac{1}{x^2} = -x^{-2}$.

□

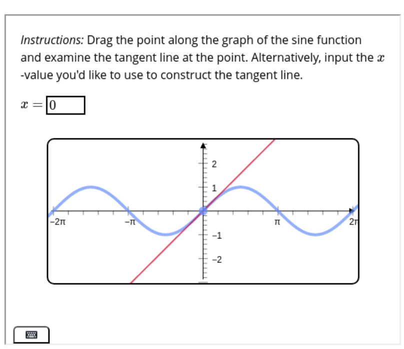
In this activity, we also found one other result.

Theorem 2.3.3 Derivative of a Constant Function. *If $y = k$ where k is some real number constant, then $y' = 0$. Another way of saying this is:*

$$\frac{d}{dx}(k) = 0.$$

Activity 2.3.2 Derivatives of Trigonometric Functions. Let's try to think through the derivatives of $y = \sin(\theta)$ and $y = \cos(\theta)$. In this activity, we'll look at graphs and try to collect some information about the derivative functions. We'll be practicing out interpretations, so if you need to brush up on Section 2.2 before we start, that's fine!

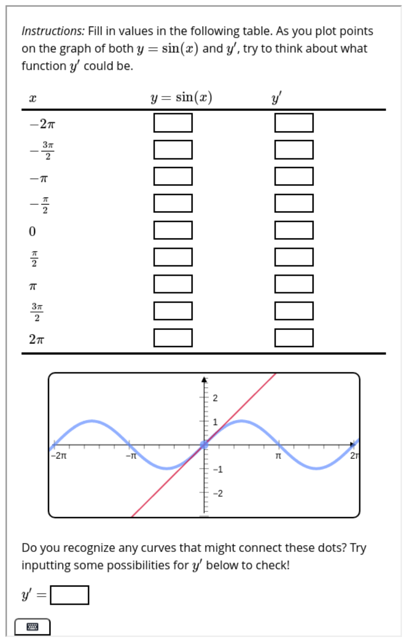
(a) The following plot includes both the graph of $y = \sin(x)$, and the line tangent to $y = \sin(x)$. Watch as the point where we build the tangent line moves along the graph, between $x = -2\pi$ and $x = 2\pi$.



Collect as much information about the derivative, y', as you can. What do you know about it? What are some facts about the slopes of the tangent lines in this animation?

Hint. What kinds of values do the slopes take? Are there some values that these slopes will never be? Can you find any special points on this graph where you can actually tell what the slope is?

(b) We're going to get more specific here: let's find the coordinates of points that are on both the graph of $y = \sin(x)$ and it's derivative y'. Remember, to get the values for y', we're really looking at the slope of the tangent line at that point.



Instructions: Fill in values in the following table. As you plot points on the graph of both $y = \sin(x)$ and y', try to think about what function y' could be.

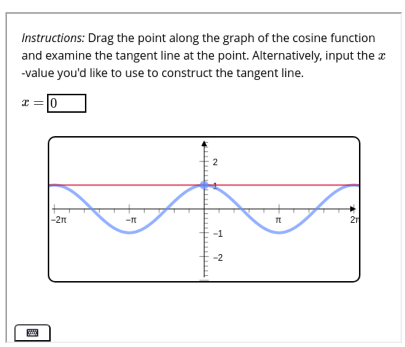
x	$y = \sin(x)$	y'
-2π		
$-\frac{3\pi}{2}$		
$-\pi$		
$-\frac{\pi}{2}$		
0		
$\frac{\pi}{2}$		
π		
$\frac{3\pi}{2}$		
2π		

Do you recognize any curves that might connect these dots? Try inputting some possibilities for y' below to check!

$y' =$

(c) Let's repeat this process using the $y = \cos(x)$ function instead.

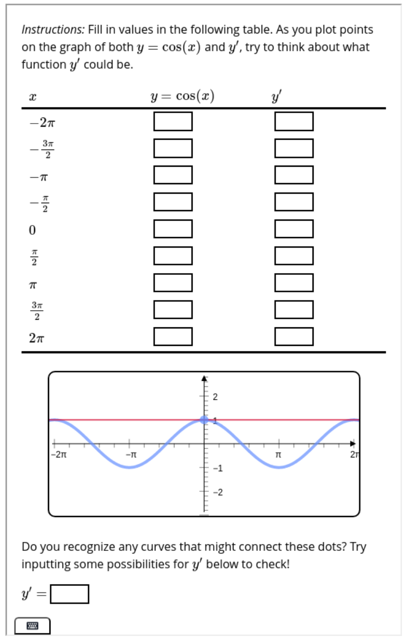
The following plot includes both the graph of $y = \cos(x)$, and the line tangent to $y = \cos(x)$. Watch as the point where we build the tangent line moves along the graph, between $x = -2\pi$ and $x = 2\pi$.



Collect as much information about the derivative, y', as you can. What do you know about it? What are some facts about the slopes of the tangent lines in this animation?

Hint. What kinds of values do the slopes take? Are there some values that these slopes will never be? Can you find any special points on this graph where you can actually tell what the slope is?

(d) We're going to get more specific here: let's find the coordinates of points that are on both the graph of $y = \cos(x)$ and it's derivative y'. Remember, to get the values for y', we're really looking at the slope of the tangent line at that point.



Instructions: Fill in values in the following table. As you plot points on the graph of both $y = \cos(x)$ and y', try to think about what function y' could be.

x	$y = \cos(x)$	y'
-2π		
$-\frac{3\pi}{2}$		
$-\pi$		
$-\frac{\pi}{2}$		
0		
$\frac{\pi}{2}$		
π		
$\frac{3\pi}{2}$		
2π		

Do you recognize any curves that might connect these dots? Try inputting some possibilities for y' below to check!

$y' =$

Theorem 2.3.4 Derivatives of the Sine and Cosine Functions.

$$\frac{d}{d\theta}(\sin(\theta)) = \cos(\theta)$$

$$\frac{d}{d\theta}(\cos(\theta)) = -\sin(\theta)$$

Proof. In order to show why $\frac{d}{d\theta}(\sin(\theta)) = \cos(\theta)$ and $\frac{d}{dx}(\cos(\theta)) = -\sin(\theta)$, we will work with the limit definitions of both. Consider both:

$$\frac{d}{d\theta}(\sin(\theta)) = \lim_{\Delta\theta \to 0}\left(\frac{\sin(\theta+\Delta\theta)-\sin(\theta)}{\Delta\theta}\right)$$
$$\frac{d}{d\theta}(\cos(\theta)) = \lim_{\Delta\theta \to 0}\left(\frac{\cos(\theta+\Delta\theta)-\cos(\theta)}{\Delta\theta}\right)$$

Our goal is to re-write the numerators in both of these limits as something more usable. So far, we have been evaluating these types of limits (First Indeterminate Forms) using some algebraic manipulations. Instead of using algebra, we will use geometry.

Consider the unit circle below. We have plotted the angle θ and are reminded that the point on the circle that corresponds with the terminal side of the angle θ has coordinates $(\cos(\theta), \sin(\theta))$. We can label the sides of the triangle pictured below.

Now we consider a second point on the circle, this one formed by the terminal side of the angle $(\theta+\Delta\theta)$. This point has coordinates $(\cos(\theta+\Delta\theta), \sin(\theta+\Delta\theta))$. Note, below, that we want to find expressions for $\sin(\theta+\Delta\theta) - \sin(\theta)$ and $\cos(\theta+\Delta\theta) - \cos(\theta)$. We can find these geometrically.

Note, then, that the two triangles look to be similar triangles. In fact, we will find that in the limit as $\Delta\theta \to 0$, the length h matches the arc length $\Delta\theta$ perfectly, and thus lays at a right angle to the terminal side of the angle $\theta+\Delta\theta$.

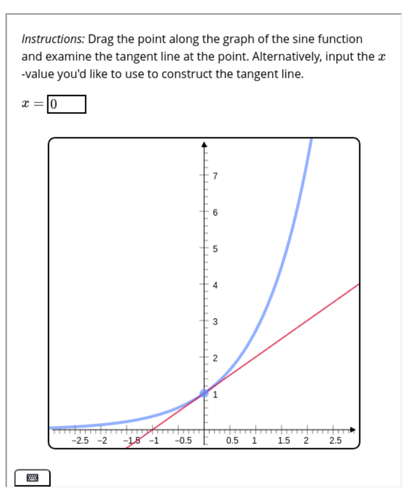
Since as $\Delta\theta \to 0$ we have $h \to \Delta\theta$, we can find the other side lengths as well: $(\sin(\theta + \Delta\theta) - \sin(\theta)) \to \Delta\theta \cos\theta$ and $(\cos(\theta + \Delta\theta) - \cos(\theta)) \to \Delta\theta \sin\theta$. So then $(\cos(\theta + \Delta\theta) - \cos(\theta)) \to -\Delta\theta \sin\theta$.
Consider, then, the limits involved in the derivative calculations that we built earlier.

$$\begin{aligned}
\frac{d}{d\theta}(\sin(\theta)) &= \lim_{\Delta\theta \to 0} \left(\frac{\sin(\theta + \Delta\theta) - \sin(\theta)}{\Delta\theta} \right) \\
&= \lim_{\Delta\theta \to 0} \left(\frac{\Delta\theta \cos(\theta)}{\Delta\theta} \right) \\
&= \lim_{\Delta\theta \to 0} (\cos(\theta)) \\
&= \cos(\theta) \\
\frac{d}{d\theta}(\cos(\theta)) &= \lim_{\Delta\theta \to 0} \left(\frac{\cos(\theta + \Delta\theta) - \cos(\theta)}{\Delta\theta} \right) \\
&= \lim_{\Delta\theta \to 0} \left(\frac{-(\cos(\theta) - \cos(\theta + \Delta\theta))}{\Delta\theta} \right) \\
&= \lim_{\Delta\theta \to 0} \left(\frac{-\Delta \sin(\theta)}{\Delta\theta} \right) \\
&= \lim_{\Delta\theta \to 0} (-\sin(\theta)) \\
&= -\sin(\theta)
\end{aligned}$$

So we have shown that $\frac{d}{d\theta}(\sin(\theta)) = \cos(\theta)$ and $\frac{d}{d\theta}(\cos(\theta)) = -\sin(\theta)$ as we claimed. ■

Activity 2.3.3 Derivative of the Exponential Function. We're going to consider a maybe-unfamiliar function, $f(x) = e^x$. We'll explore this function in a similar way to use thinking about the derivatives of sine and cosine in Activity 2.3.2: we'll look at a tangent line at different points, and think about the slope.

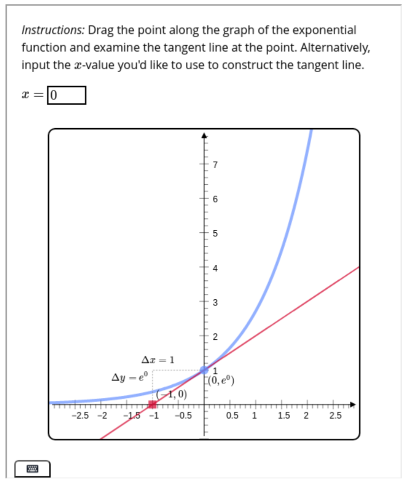
(a) The plot below includes both the graph of $y = e^x$ and the line tangent to $y = e^x$. Watch as the point moves along the curve.



Collect as much information about the derivative, y', as you can. What do you know about it? What are some facts about the slopes of the tangent lines in this animation?

Hint. Are there any x-values where the slope is negative? Are there any where the slope is equal to 0? What happens to the slopes as x increases?

(b) There is a slightly hidden fact about slopes and tangent lines in this animation. In the following animation, we'll add (and label) one more point. Let's look at this again, this time noting the point at which this tangent line crosses the x-axis. This will make it easier to think about slopes!



What information does this reveal about the slopes?

Hint. Especially it might be helpful to think about the slopes and their relationship to the y-value of the point we are building the tangent line at.

(c) Make a conjecture about the slope of the line tangent to the exponential function $y = e^x$ at any x-value. What do you believe the formula/ equation for y' is then?

Theorem 2.3.5 Derivative of the Exponential Function.

$$\frac{d}{dx}\left(e^x\right) = e^x$$

2.3.2 Some Properties of Derivatives in General

Theorem 2.3.6 Combinations of Derivatives. *If $f(x)$ and $g(x)$ are differentiable functions, then the following statements about their derivatives are true.*

1. Sums: *The derivative of the sum of $f(x)$ and $g(x)$ is the sum of the derivatives of $f(x)$ and $g(x)$:*

$$\begin{aligned}\frac{d}{dx}\left(f(x)+g(x)\right) &= \left(\frac{d}{dx}f(x)\right) + \left(\frac{d}{dx}g(x)\right)\\ &= f'(x) + g'(x)\end{aligned}$$

2. Differences: *The derivative of the difference of $f(x)$ and $g(x)$ is the difference of the derivatives of $f(x)$ and $g(x)$:*

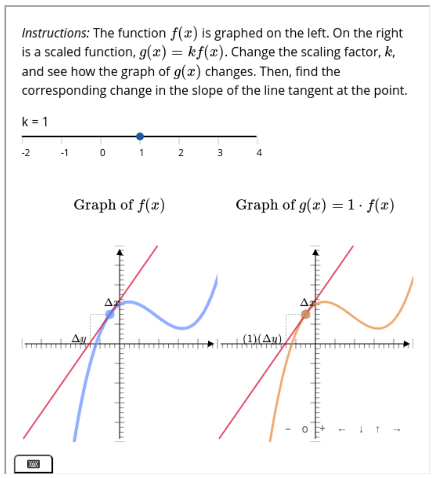
$$\frac{d}{dx}\left(f(x)-g(x)\right) = \left(\frac{d}{dx}f(x)\right) - \left(\frac{d}{dx}g(x)\right)$$

$$= f'(x) - g'(x)$$

3. Coefficients: *If k is some real number coefficient, then:*

$$\frac{d}{dx}\left(kf(x)\right) = k\left(\frac{d}{dx}f(x)\right)$$
$$= kf'(x)$$

We can think about each of these properties through the lense of how combining these functions impacts the slopes. For instance, if we wanted to visualize the property about coefficients (that $\frac{d}{dx}\left(kf(x)\right) = k\frac{d}{dx}\left(f(x)\right)$), we can visualize this coefficient as a scaling factor.



Example 2.3.7 Putting These Together. Find the following derivatives:

(a) $\frac{d}{dx}\left(4x^5 - \frac{5x}{2} + 6\cos(x) - 1\right)$

Solution.

$$\begin{aligned}\frac{d}{dx}\left(4x^5 - \frac{5x}{2} + 6\cos(x) - 1\right) &= \frac{d}{dx}\left(4x^5\right) - \frac{d}{dx}\left(\frac{5x}{2}\right) + \frac{d}{dx}\left(6\cos(x)\right) - \frac{d}{dx}(1)\\ &= 4\frac{d}{dx}\left(x^5\right) - \frac{5}{2}\frac{d}{dx}(x) + 6\frac{d}{dx}\left(\cos(x)\right) - \frac{d}{dx}(1)\\ &= 4(5x^4) - \frac{5}{2}(1) + 6(-\sin(x)) - 0\\ &= 20x^4 - \frac{5}{2} - 6\sin(x)\end{aligned}$$

□

2.4 The Product and Quotient Rules

We saw in Theorem 2.3.6 Combinations of Derivatives that when we want to find the derivative of a sum or difference of functions, we can just find the derivatives of each function separately, and then combine the derivatives back together (by adding or subtracting). This, hopefully, is pretty intuitive: of course a slope of a sum of things is just the slopes of each thing added together!

In this section, we want to think about derivatives of product and quotients of functions. What happens when we differentiate a function that is really just two functions multiplied together?

Activity 2.4.1 Thinking About Derivatives of Products. Let's start with two quick facts:

$$\frac{d}{dx}\left(x^3\right) = 3x^2 \text{ and } \frac{d}{dx}\left(\sin(x)\right) = \cos(x).$$

(a) What is $\frac{d}{dx}\left(x^3 + \sin(x)\right)$? What about $\frac{d}{dx}\left(x^3 - \sin(x)\right)$?

(b) Based on what you just explained, what is a reasonable assumption about what $\frac{d}{dx}\left(x^3 \sin(x)\right)$ might be?

Hint. Does it seem reasonable that we could just multiply the derivatives together, and say that $\frac{d}{dx}\left(x^3 \sin(x)\right)$ was the same thing as

$$\frac{d}{dx}\left(x^3\right) \cdot \frac{d}{dx}\left(\sin(x)\right)?$$

(c) Let's just think about $\frac{d}{dx}\left(x^3\right)$ for a moment. What *is* x^3? Can you write this as a product? Call one of your functions $f(x)$ and the other $g(x)$. You should have that $x^3 = f(x)g(x)$ for whatever you used.

(d) Use your example to explain why, in general, $\frac{d}{dx}\left(f(x)g(x)\right) \neq \frac{d}{dx}\left(f(x)\right) \cdot \frac{d}{dx}\left(g(x)\right)$.

(e) Let's show another way that we know this. Think about $\sin(x)$. We know two things:

$$\sin(x) = (1)(\sin(x)) \text{ and } \frac{d}{dx}\left(\sin(x)\right) = \cos(x).$$

What, though, is $\frac{d}{dx}(1) \cdot \frac{d}{dx}\left(\sin(x)\right)$?

(f) Use all of this to reassure yourself that even though the derivative of a sum of functions is the sum of the derivatives of the functions, we will need to develop a better understanding of how the derivatives of products of functions work.

A thing that I like to think about is this: if $\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g'(x)$, then every function's derivative would be 0.

In the example with the $\sin(x)$ function, we noticed that we could write the function as $(1)(\sin(x))$. This is true of every function!

If $\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g'(x)$, then we could say that for any function $f(x)$ with a derivative $f'(x)$:

$$\begin{aligned}\frac{d}{dx}\left(f(x)\right) &= \frac{d}{dx}\left(1 \cdot f(x)\right)\\ &= \frac{d}{dx}(1)\,\frac{d}{dx}\left(f(x)\right)\\ &= 0 \cdot f'(x)\\ &= 0.\end{aligned}$$

This, obviously, can't be true! We know of *tons* of functions that have non-zero slopes...*most* of them do!

So, we hopefully have some motivation for building a rule to that helps us think about derivatives of products of functions.

2.4.1 The Product Rule

Activity 2.4.2 Building a Product Rule. Let's investigate how we might differentiate the product of two functions:

$$\frac{d}{dx}\left(f(x)g(x)\right).$$

We'll use an area model for multiplication here: we can consider a rectangle where the side lengths are functions $f(x)$ and $g(x)$ that change for different values of x. Maybe x is representative of some type of time component, and the side lengths change size based on time, but it could be anything.

If we want to think about $\frac{d}{dx}\left(f(x)g(x)\right)$, then we're really considering the change in area of the rectangle.

(a) Find the area of the two rectangles. The second rectangle is just one where the input variable for the side length has changed by some amount, leading to a different side length.

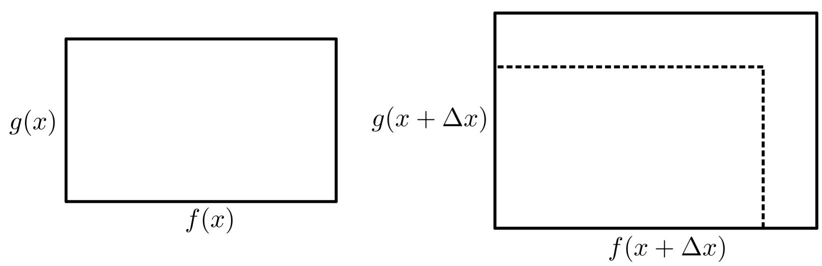


Figure 2.4.1

(b) Write out a way of calculating the difference in these areas.

(c) Let's try to calculate this difference in a second way: by finding the actual area of the region that is new in the second rectangle.

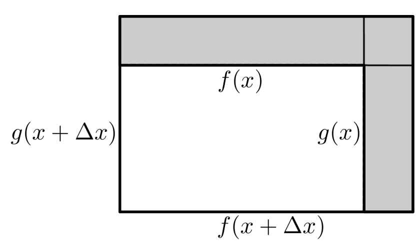


Figure 2.4.2

In order to do this, we've broken the region up into three pieces. Calculate the areas of the three pieces. Use this to fill in the following equation:

$f(x+\Delta x)g(x+\Delta x)-f(x)g(x) =$ __________.

(d) We do not want to calculate the total change in area: a derivative is a *rate of change*, so in order to think about the derivative we need to divide by the change in input, Δx.

We'll also want to think about this derivative as an *instantaneous* rate of change, meaning we will consider a limit as $\Delta x \to 0$. Fill in the following:

$$\frac{d}{dx}\left(f(x)g(x)\right) \lim_{\Delta x \to 0}\left(\frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}\right)$$
$$= \lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right)$$

We can split this fraction up into three fractions:

$$\frac{d}{dx}\left(f(x)g(x)\right) = \lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right) + \lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right) + \lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right)$$

(e) In any limit with $f(x)$ or $g(x)$ in it, notice that we can factor part out of the limit, since these functions do not rely on Δx, the part that changes in the limit. Factor these out.

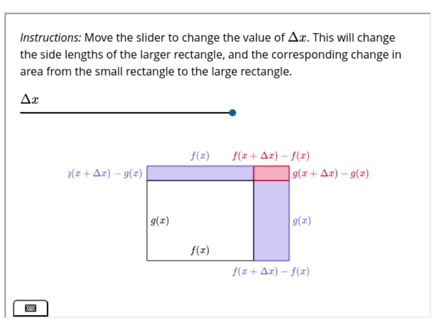
In the third limit, factor out either $\lim_{\Delta x \to 0}\left(f(x+\Delta x) - f(x)\right)$ or $\lim_{\Delta x \to 0}\left(g(x+\Delta x) - g(x)\right)$.

$$\frac{d}{dx}\left(f(x)g(x)\right) = f(x)\lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right) + g(x)\lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right) + \lim_{\Delta x \to 0}\left(\boxed{}\right)\left(\lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right)\right)$$

(f) Use Definition 2.1.2 The Derivative Function to re-write these limits. Show that when $\Delta x \to 0$, we get:

$$f(x)g'(x) + g(x)f'(x) + 0.$$

We can investigate this visual a bit more dynamically: see the differences in area as $\Delta x \to 0$. What parts are left, when $\Delta x \to 0$? What areas aren't visible?



Theorem 2.4.3 Product Rule. *If $u(x)$ and $v(x)$ are functions that are differentiable at x and $f(x) = u(x) \cdot v(x)$, then:*

$$\frac{d}{dx}(f(x)) = u'(x) \cdot v(x) + u(x) \cdot v'(x).$$

For convenience, this is often written as:

$$\frac{d}{dx}(u \cdot v) = u'v + uv' \qquad \text{or} \qquad \frac{d}{dx}(u \cdot v) = v\left(\frac{du}{dx}\right) + u\left(\frac{dv}{dx}\right).$$

Example 2.4.4 Use the Product Rule to find the following derivatives.

(a) $\dfrac{d}{dx}\left(x^3 \sin(x)\right)$

Hint. Use $u = x^3$ and $v = \sin(x)$. Now find u' and v', and use:

$$\frac{d}{dx}(uv) = u'v + uv'.$$

(b) $\dfrac{d}{dx}\left((x^3 + 4x)e^x\right)$

(c) $\dfrac{d}{dx}(\sqrt{x}\cos(x))$

□

2.4.2 What about Dividing?

So we can differentiate a product of functions, and the obvious next question should be about division: if we needed to build a reasonable way of differentiating a product, shouldn't we also need to build a new way of thinking about derivatives of quotients?

A nice thing that we can do is think about division as really just multiplication. For instance, if we want to differentiate $\dfrac{d}{dx}\left(\dfrac{\sin(x)}{x^2}\right)$, then we can just think about this quotient as really a product:

$$\frac{d}{dx}\left(\frac{\sin(x)}{x^2}\right) = \frac{d}{dx}\left(\frac{1}{x^2}(\sin(x))\right).$$

Now we can just apply product rule!

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{x^2}(\sin(x))\right) &= \frac{d}{dx}\left(x^{-2}\sin(x)\right)\\
u = \sin(x) \quad & v = x^{-2}\\
u' = \cos(x) \quad & v' = -2x^{-3}\\
\frac{d}{dx}\left(\sin(x)x^{-2}\right) &= x^{-2}\cos(x) + (-2x^{-3}\sin(x))\\
&= \frac{\cos(x)}{x^2} - \frac{2\sin(x)}{x^3}
\end{aligned}$$

This works great! We can *always* think about quotients as just products of reciprocals! But the problem is that we can't always differentiate these reciprocals (for now). We were able to differentiate $\dfrac{1}{x^2}$ by re-writing this as just a power function (with a negative exponent).

What about this flipped example:

$$\frac{d}{dx}\left(\frac{x^2}{\sin(x)}\right)?$$

In order for us to do the same thing, we need to re-write this as

$$\frac{d}{dx}\left(x^2\left(\sin(x)\right)^{-1}\right)$$

but we don't know how to differentiate $\left(\sin(x)\right)^{-1}$ (yet).

So let's try to build a general way of differentiating quotients.

Activity 2.4.3 Constructing a Quotient Rule. We're going to start with a function that is a quotient of two other functions:

$$f(x) = \frac{u(x)}{v(x)}.$$

Our goal is that we want to find $f'(x)$, but we're going to shuffle this function around first. We won't calculate this derivative directly!

(a) Start with $f(x) = \dfrac{u(x)}{v(x)}$. Multiply $v(x)$ on both sides to write a definition for $u(x)$.

$u(x) =$

(b) Find $u'(x)$.

(c) Wait: we don't care about $u'(x)$. Right? We care about finding $f'(x)$! Use what you found for $u'(x)$ and solve for $f'(x)$.

$f'(x) =$

(d) This is a strange formula: we have a formula for $f'(x)$ written in terms of $f(x)$! But we said earlier that $f(x) = \dfrac{u(x)}{v(x)}$.

In your formula for $f'(x)$, replace $f(x)$ with $\dfrac{u(x)}{v(x)}$.

$f'(x) =$

This formula is fine, but a little clunky. We'll try to re-write it in some nice ways, but it is a bit more complex than the Product Rule.

Theorem 2.4.5 Quotient Rule. *If $u(x)$ and $v(x)$ are differentiable at x and $f(x) = \dfrac{u(x)}{v(x)}$ then:*

$$f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{(v(x))^2}.$$

For convenience, this is often written as:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}.$$

Example 2.4.6 Use the Quotient Rule to find the following derivatives.

(a) $\frac{d}{dx}\left(\frac{\sin(x)}{x^2}\right)$

Once you have this derivative, confirm that it is the same as $\frac{\cos(x)}{x^2} - \frac{2\sin(x)}{x^3}$, the way that we found it using the Product Rule.

(b) $\frac{d}{dx}\left(\frac{x^2}{\sin(x)}\right)$

(c) $\frac{d}{dx}\left(\frac{x+4}{x^2+1}\right)$

□

2.4.3 Derivatives of (the Rest of the) Trigonometric Functions

You might remember that of the six main trigonometric functions, we can write four of them in terms of the other two: here are the different trigonometric functions written in terms of sine and cosine functions:

$$\tan(x) = \left(\frac{\sin(x)}{\cos(x)}\right)$$
$$\sec(x) = \left(\frac{1}{\cos(x)}\right)$$
$$\cot(x) = \left(\frac{\cos(x)}{\sin(x)}\right)$$
$$\csc(x) = \left(\frac{1}{\sin(x)}\right)$$

Example 2.4.7 Find the derivatives of the remaining trigonometric functions.

(a) $\frac{d}{dx}(\tan(x))$

Hint. Write $\frac{d}{dx}(\tan(x)) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)$ and use the Quotient Rule.

(b) $\frac{d}{dx}(\sec(x))$

Hint. Write $\frac{d}{dx}(\sec(x)) = \frac{d}{dx}\left(\frac{1}{\cos(x)}\right)$ and use the Quotient Rule.

(c) $\frac{d}{dx}(\cot(x))$

Hint. Write $\frac{d}{dx}(\cot(x)) = \frac{d}{dx}\left(\frac{\cos(x)}{\sin(x)}\right)$ and use the Quotient Rule.

(d) $\frac{d}{dx}(\csc(x))$

Hint. Write $\frac{d}{dx}(\csc(x)) = \frac{d}{dx}\left(\frac{1}{\cos(x)}\right)$ and use the Quotient Rule.

□

2.5 The Chain Rule

We've been building up some intuition and rules to help us think about differentiating different functions and combinations of functions. We can find derivatives of scaled functions, sums of functions, differences of functions, products of functions, and also quotients of functions.

In this section, we'll look at our last operation between functions: composition.

2.5.1 Composition and Decomposition

An important part of finding derivatives of products and quotients is identifying the component functions that are being multiplied/divided (often labeled $u(x)$ or just u and $v(x)$ or just v). From there, we find the derivatives of each of the component functions, and then use the formula from the Product Rule or Quotient Rule to put the pieces together.

Thinking about derivatives of composed functions will be the same: we'll need to identify what functions are being composed inside of other functions, and use those pieces in some formulaic way to represent the derivative. On that note, let's remind ourselves and practice working with composition (and decomposition) of functions.

Activity 2.5.1 Composition (and Decomposition) Pictionary. This activity will involve a second group, or at least a partner. We'll go through the first part of this activity, and then connect with a second group/person to finish the second part.

(a) Build two functions, calling them $f(x)$ and $g(x)$. Pick whatever kinds of functions you'd like, but this activity will work best if these functions are in a kind of sweet-spot between "simple" and "complicated," but don't overthink this.

(b) Compose $g(x)$ inside of $f(x)$ to create $(f \circ g)(x)$, which we can also write as $f(g(x))$.

(c) Write your composed $f(g(x))$ function on a separate sheet of paper. Do not leave any indication of what your chosen $f(x)$ and $g(x)$ are. Just write your composed function by itself.

Now, pass this composed $f(g(x))$ to your partner/a second group.

(d) You should have received a new function from some other person/group. It is different than yours, but also labeled $f(g(x))$ (with different choices of $f(x)$ and $g(x)$).

Identify a possibility for $f(x)$, the outside function in this composition, as well as a possibility for $g(x)$, the inside function in this composition. You can check your answer by composing these!

(e) Write a different pair of possibilities for $f(x)$ and $g(x)$ that will still give you the same composed function, $f(g(x))$.

(f) Check with your partner/the second group: did you identify the pair of functions that they originally used?

Did whoever you passed your composed function to correctly identify your functions?